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Quantum Probabilities, Operators of State Preparation, and the Principle of Superposition

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An integrated view concerning the probabilistic organization of quantum mechanics is first obtained by systematic confrontation of the Kolmogorov formulation of the abstract theory of probabilities with the quantum mechanical representation *and* its factual counterparts. Because these factual counterparts possess a peculiar space-time structure stemming from the operations by which the observer produces the studied states (operations of state preparation) and the qualifications of these (operations of measurement), the approach brings forth "probability-trees," complex constructs with treelike space-time support. Though it is strictly entailed by confrontation with the abstract theory of probabilities as it now stands, the construct of a quantum mechanical probability tree *transgresses* this theory. It indicates the possibility of an extended abstract theory of probabilities: Quantum mechanics appears to be neither a "normal" probabilistic theory nor an "abnormal" one, but a pioneering particular realization of a *future* extended abstract theory of probabilities. The integrated perception of the probabilistic organization of quantum mechanics removes the current identifications of spectral decompositions of one state vector, with superpositions of several state vectors. This leads to the definition of operators of state preparation and of the calculus with these and to a clear understanding of the physical significance of the principle of superposition. Furthermore, a complement to the quantum theory of measurements is obtained.

1. PRELIMINARY

In this work we combine results already exposed in other works (Mugur-Schächter, 1983, 1984, 1985, 1991, 1992*a,b*). But the lighting is new, concentrated upon the definition of operators of state preparation and the physical significance of the principle of superposition.

Sometimes the exposition reproduces *ad literam* previously published texts. This reflects the fact that in the present stage of the development of

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our views the expression of the ideas that we try to convey seems to us to have reached a status of equilibrium (or optimality) which we are reluctant to disturb.

2. THE QUANTUM MECHANICAL PROBABILITY TREES

2.1. The Abstract Theory of Probabilities, Physical Probabilistic Theories, Quantum Mechanics

In Kolmogorov's formulation of the abstract theory of probabilities any probability measure π is defined inside a probability space $[\mathrm{U}, \tau, \pi]$, where $\mathrm{U}=\{e_i\}$ ($i \in I$, I an index set) is a universe of elementary events e_i, τ is an algebra of events chosen on U, and π is a probability measure posed on τ. Furthermore, the universe U is conceived to be produced by a random phenomenon. But currently this supposed random phenomenon is neither defined nor symbolized. Throughout what follows this lacuna will be compensated: Let us denote a random phenomenon by (P, U), where P is an "identically" reproducible procedure, each realization of which brings forth one elementary event $e_i \in \mathrm{U}$, in general variable from one realization of P to another one (notwithstanding the supposed identity of the reiterations), whereby the whole universe U is generated. In order to express explicitly that each probability space is tied to some random phenomenon, we shall always consider a complete "probability *chain*" where the probability space is preceded by the explicit symbolization of the corresponding random phenomenon:

$$(P, \mathrm{U}) \rightsquigarrow [\mathrm{U}, \tau, \pi] \tag{1}$$

The abstract theory of probabilities does not describe specified phenomena, it only introduces symbols and defines the calculi with these characterizing any probabilistic conceptualization of phenomena of any nature. As soon as some specified domain of reality undergoes a probabilistic conceptualization, an interpretation of the abstract theory is obtained. Inside this interpretation, unavoidably, some probability chains are supposed, but where now the constituting symbols point more or less explicitly toward entities from the described domain of reality. So *a particular semantics* comes in. But very often when physical problems are treated probabilistically only the probability measures are defined explicitly and are symbolized. The elementary events and the algebra of events are usually indicated by words only, while currently the random phenomenon which produces them remains entirely implicit. However, by reference to the abstract theory of probabilities, it is obvious that without a universe of elementary events, without an algebra of events chosen on this universe, a probability measure simply is not defined. It does not conceptually exist. A probability measure alone is not a concept, it is a rag of a concept. Furthermore, by definition, in the absence of any random phenomenon, a universe of elementary events cannot emerge, hence no probability space either: The probability chains (1) are *indivisible molds* imposed by the abstract theory of probabilities. So what are the particular probability chains specific of quantum mechanics? What is the specific semantics toward which point the quantum mechanical probability chains?

2.2. The Quantum Mechanical Representation of the Probabilistic Aspects from the Theory

For the sake of simplicity, throughout what follows we consider exclusively the basic case of only "one microsystem," whatever definition one associates to this concept. This will suffice for conveying the essence of our view.

2.2.1. The Formal Quantum Mechanical Probability Chains

Consider a pair $(|\psi\rangle, \mathrm{A})$ where $|\psi\rangle = |\psi(t)\rangle$ is the state vector assigned at the time t to the considered microsystem S, and A is a Hermitian operator representing a dynamical observable—in the mathematical sense—defined for S. For each such pair the quantum mechanical formalism defines a family of probability densities $\pi(\psi, a_j)$, $j \in J$ (J an index set) for the emergence of an eigenvalue a_j of the observable A when a measurement of A is performed on S in the state $|\psi\rangle$. Namely, it is postulated that the specified probability density can be calculated by use of the formula (for simplicity we suppose a nondegenerate situation)

$$\forall j \in J, \qquad \pi(\psi, a_j) = |\langle u_j | \psi \rangle|^2 \tag{2}$$

where $|u_j\rangle$ is the eigenvector corresponding to the considered eigenvalue a_j, determined, like a_j, by the equation $\mathrm{A}|u_j\rangle = a_j|u_j\rangle$ for eigenvectors and eigenvalues of A. Usually the algorithm (2) for the computation of probability measures is postulated without any explicit specification of the probability space where the measure (2) is incorporated, nor, *a fortiori*, of the random phenomenon from which this space stems. But it is obvious that the space which contains the measure (2) can be represented by

$$[\mathbf{a}, \tau_{\mathrm{A}}, \pi(\psi, \mathrm{A})] \tag{3}$$

where the universe of elementary events $\mathbf{a} = \{a_j, j \in J\}$ (J an index set) is the spectrum of the observable A, τ_{A} is the total algebra of events on $\mathbf{a}$, and $\pi(\psi, \mathrm{A})$ is the probability density measure on τ_{A} determined, via the law of total probabilities, by the elementary probability density (2). So the whole probability *chain* corresponding to a space (3) can be represented by the writing

$$(|\psi\rangle, \mathrm{A}) \rightsquigarrow [\mathbf{a}, \tau_{\mathrm{A}}, \pi(\psi, \mathrm{A})] \qquad (1')$$

This is the sought integrated representation of the formal quantum mechanical probability-chains, achieved with the help of the quantum mechanical descriptors.

2.2.2. *The Factual Quantum Mechanical Probability Chains*

The formal chains (1′) are only a coded representation of other, *factual* quantum mechanical probability chains. Let us now identify these factual chains.

The Factual Quantum Mechanical Probability Spaces. We postpone the specification of the factual random phenomenon corresponding to the symbol $(|\psi\rangle, \mathrm{A})$ from the chain (1′) and we consider first only the space (3), $[\mathbf{a}, \tau_{\mathrm{A}}, \pi(\psi, \mathrm{A})]$ involved in this chain. The corresponding factual space can be immediately specified as follows:

$$[\mathrm{V}_A(\mathrm{D}_A, t_2), \tau_A, \pi(\psi, \mathrm{M}_A)] \qquad (3')$$

where A designates an observable and numerically valued physical aspect *of a macroscopic device* D_A able to generate certain materializations of the numerical values to be assigned to the quantum mechanical observable (in the mathematical sense this time) A, namely "needle positions" of D_A; $\mathrm{V}_A(\mathrm{D}_A, t_A)$ is the universe of all the possible values V_j of the physical aspect A of D_A, a universe brought forth by "one" realization of what is *globally* called a "measurement process" of the observable A, consisting by definition of a very big number of *reiterations* of a registration of a value V_j, operated each time by starting from the state of S symbolized by the state vector $|\psi\rangle$ *newly prepared* and each such registration covering some spatial domain d_A and beginning at a time t when the state vector of S is $|\psi\rangle$ and then *lasting* for some nonnull time interval $(t_A - t) > 0$ (let us denote this measurement process by $\mathrm{M}_A(\psi, \mathrm{D}_A)$); τ_A is the total algebra on the universe $\mathrm{V}_A(\mathrm{D}_A, t_A)$; and $\pi(\psi, \mathrm{M}_A)$ is the density of the probability measure put on τ_A, depending on the state labeled by the state vector $|\psi\rangle$ and on the measurement process M_A performed on this state.

The probability measure $\pi(\psi, \mathrm{M}_A)$ on the algebra τ_A from the probability space (3′) is determined, via the law of total probabilities, by the probability density $\pi(\psi, \mathrm{M}_A, V_j)$ postulated on the universe $\mathrm{V}_A(\mathrm{D}_A, t_A) = \{V_j, j \in J\}$ of elementary events from this space.

The Factual Quantum Mechanical Random Phenomena. What is the factual random phenomenon that brings forth the universes of elementary events $V_A(D_A, t_A) = \{V_j, j \in J\}$ from a factual quantum mechanical probability space (3′)? This random phenomenon possess a complex structure. It brings in a sequence of three partial procedures covering three distinct space-time domains:

1. The first partial procedure is the preparation operation $P(\psi_0)$ which, at its final moment t_0 (supposed to be definable), introduces an initial state of S represented by the state vector $|\psi(t_0)\rangle = |\psi_0\rangle$; this operation covers some nonnull space-time domain $[\Delta\mathbf{r} \times \Delta t_0]_P$.

2. The second partial procedure, which does not necessarily exist, is a process $E(H, t_0, t)$ of evolution of the initial state of S, leading at the time t to the state with state vector $|\psi(t)\rangle = |\psi\rangle$. When it does exist, this evolution (formally described by the writing $|\psi\rangle = T(H, t_0, t)|\psi_0\rangle$, where $T(H, t_0, t)$ is the acting propagator) covers some new space-time interval $[\Delta\mathbf{r} \times \Delta t]_E$, where $\Delta t = t - t_0$.

3. The third partial procedure is the measurement operation $M_A(\psi, D_A)$ from the definition of the observable space (3′), performed on the state of S symbolized by the state vector $|\psi\rangle$.

As soon as the time $t \geq t_0$ is fixed, the succession

$$P = [P(\psi_0), E(H, t_0, t), M_A(\psi, D_A)] \tag{4}$$

constitutes "*one* identically reproducible procedure P," each reiteration of P reestablishing the origin of times t_0. Note that the succession of only the first two partial procedures from (4) can be regarded as a preparation operation $P(\psi)$ producing the *studied* state represented by the state vector $|\psi\rangle = T(H, t_0, t)|\psi_0\rangle$. So we can also write

$$P = [P(\psi), M_A(\psi, D_A)] \tag{4'}$$

where the initial operation $P(\psi_0)$ and the evolution symbolized by $T(H, t_0, t)$ become implicit.

Each realization of the procedure P brings forth *one*, V_j, among all the various possible elementary events from the universe of elementary events $U = V_A$. Thus we are finally in presence of a random phenomenon (P, U) in the *standard* sense of the term, namely

$$(P, U) = ([P(\psi_0), E(H, t_0, t), M_A(\psi, D_A)], V_A(D_A, t_A)) \tag{5}$$

or

$$(P, U) = ([P(\psi), M_A(\psi, D_A)], V_A(D_A, t_A)) \tag{5'}$$

The Factual Quantum Mechanical Probability Chains. So the factual quantum mechanical probability chains can be written as follows:

$$([\mathrm{P}(\psi), \mathrm{M}_A(\psi, \mathrm{D}_A)], \mathrm{V}_A(\mathrm{D}_A, t_A)) \rightsquigarrow [\mathrm{V}_A(\mathrm{D}_A, t_A), \tau_A, \pi(\psi, \mathrm{M}_A)] \quad (1'')$$

The expressions (3′) to (1″) indicate now explicitly and exhaustively the specific semantic contents of the quantum mechanical probability chains.

2.2.3. The Connection between the Factual and the Formal Quantum Mechanical Probability Spaces

How can we translate a factual observable quantum mechanical probability space into the corresponding formal space so as to be able to apply to it the quantum mechanical algorithms?

In quantum mechanics each eigenvalue $a_j \in \mathbf{a}$ is posited to be calculable as a function $f_\mathrm{A}(V_j)$ of the observed factual value $V_j \in \mathrm{V}_A(\mathrm{D}_A, t_2)$ which is labeled by the same index $j \in J$:

$$a_j = f_\mathrm{A}(V_j) \quad (6)$$

Furthermore, each observable elementary probability density $\pi(\psi, \mathrm{M}_A, V_j)$ is posed to be numerically equal to the corresponding formal elementary probability density, i.e., for any $|\psi\rangle$ and any $j \in J$, it is postulated that (degenerate cases being excluded)

$$\pi(\psi, \mathrm{M}_A, V_j) = \pi(\psi, a_j) = |\langle u_j|\psi\rangle|^2 \quad (7)$$

where $|u_j\rangle$ is the eigenvector of the observable A corresponding to the eigenvalue $a_j = f_\mathrm{A}(V_j)$. [Notice that thereby a_j can be regarded as a random variable on the factual space (3′), a space that is not defined inside the formalism.] In *this* sense, the formal probability density (2) is a "predictional law," verifiable with the help of the relative frequencies of emergence of the observed values V_j, at the limit of large numbers.

Equations (6) and (7) form the key of the code which translates the factual observable quantum mechanical probability space (3′) into the formal space (3). Any quantum mechanical prediction belongs to some formal probability space (3) corresponding to a factual space (3′).

2.2.4. The Processual Roots of the Quantum Mechanical Elementary Events in the Sense of Probabilities

The expression (5) of a factual quantum mechanical random phenomenon involves reiterations of a chain of operations and processes:

[(preparation operation $\mathrm{P}(\psi_0)$)-(evolution process E)-(measurement operation M_A)-(registration of a needle position V_j of the utilized device D_A)] (eqmce)

[(eqmce): elementary quantum mechanical chain experiment]: *These are the processual roots of the quantum mechanical elementary events in the sense of probabilities.*

An elementary quantum mechanical chain experiment possesses a remarkable unobservable *depth* wherefrom there emerges into the observable only the extremity $V_j, j \in J$, that contributes to the construction of the factual observable universe of elementary events $\mathrm{V}_A(\mathrm{D}_A, t_A) = \{V_j, j \in J\}$. Each observable quantum mechanical "event" (*non*elementary) from an algebra τ_A from a factual quantum mechanical probability space (3′) contains inside its semantic substratum all the unobservable chains of operations and processes forming the elementary quantum mechanical chain experiments that end up with the registration of a needle position V_j contained in that factual observable quantum mechanical event. So *any* quantum mechanical prediction concerns either an elementary quantum mechanical chain experiment or a union of such experiments. *The elementary quantum mechanical chain experiments (eqmce) yield the "fibers" out of which is made the factual substance of the quantum theory.*

2.2.5. Partial Conclusion

We are now endowed with an explicit knowledge of the relations between, on the one hand, the basic abstract concepts of the probabilistic conceptualization (identically reproducible procedure P, universe of elementary events U, algebra of events τ, probability measure π), and on the other hand, the quantum mechanical formal descriptors, state vectors $|\psi\rangle$, observables A, eigenvectors $|u_j\rangle$, eigenvalues a_j. It appears that quantum mechanics contains definite realizations of each basic concept from the present abstract theory of probabilities. So, in *this* sense, it can be asserted that quantum mechanics is *not* an "abnormal" probabilistic theory. Furthermore, we have also explicated the specific semantical content assigned by the quantum mechanical description to the basic abstract probabilistic concepts. Now, do these first results entail that quantum mechanics is a "normal" probabilistic theory?

2.3. The Probability Trees of State Preparations

We arrive now at the crucial point of this section, where *new* consequences of the preceding analysis will manifest themselves.

We have shown that any quantum mechanical prediction concerns one or several elementary quantum mechanical chain experiments. We shall now show that the ensemble of all the elementary quantum mechanical chain experiments falls into *classes of metastructures* possessing a treelike space-time organization.

Let us *fix* a preparation $P(\psi_0)$, a time interval $\Delta t = t - t_0$, and a Hamiltonian H. Consider now the ensemble of all the probability chains (5) or (5′) corresponding to the fixed pair $(P(\psi_0), |\psi\rangle)$ *and* to *all* the distinct dynamical observables A, B, C, D, . . . defined in quantum mechanics: The chains from this ensemble constitute together a certain *unity*, because of their common provenance $(P(\psi_0), |\psi\rangle)$. What is the space-time structure of this unity?

For all the chains from the considered unity, the space-time support of the operation of state preparation $P(\psi_0)$ and of the Schrödinger evolution $T(H, t_0, t)|\psi_0\rangle = |\psi\rangle$, $t \geq t_0$, of the prepared state which follows this operation is, by construction, the *same*, a common space-time trunk. If in particular $|\psi\rangle \equiv |\psi_0\rangle$, i.e., if $t = t_0$, then the trunk is reduced to the operation of state preparation alone.

Consider now the space-time supports of the measurement processes M_A involved in this unity. The ensemble of these processes *splits* into subensembles M_X, M_Y, . . . of mutually "compatible" processes of "measurement evolution" corresponding to mutually commuting observables.

Contrary to many very confusing considerations concerning "successive measurements of compatible observables" (versus the projection postulate) that can be currently found in the textbooks of quantum mechanics, let us emphasize this: Each *one* measurement evolution from the subensemble M_X is such that each registration of a value V_j of the "needle position" of the macroscopic device D_X associated with M_X permits us to calculate, from the *unique* datum V_j, via a set of *various* theoretical connecting definitions (6), $a_j = f_A(V_j)$, $b_j = f_B(V_j)$, . . . , *all* the *different* eigenvalues a_j, b_j, . . . , labeled by the *same* index j, for, respectively, all the observables A, B, . . . , measurable by a process belonging to the class M_X. This entails that *for all the commuting observables corresponding to one same class M_X, the process of registration of a value of the "needle position" of the device D_X can be* one *common process covering one common space*-time *support* (no succession whatever is necessary). While this is *not* possible for two noncommuting observables belonging to two distinct classes M_X and M_Y:

This is what is commonly designated as "Bohr complementarity," nothing else.

Now, this entails that, globally, the ensemble of all the factual probability chains (1″) corresponding to a fixed pair $(P(\psi_0), |\psi\rangle)$ constitutes a unity, a metaconstruct, *with a branching, treelike space-time structure*. Let us symbolize this treelike structure by $\mathcal{T}(\psi_0, \psi)$ and let us call it a "quantum mechanical probability tree" (in short, a probability tree). [Since all the probability trees involving the same studied state vector $|\psi\rangle$ introduce the same branch structure, carrying on top the same probability spaces, in contexts where the distinction between the state vector of the initially prepared state and that of the studied state is not relevant we shall assume that $|\psi\rangle \equiv |\psi_0\rangle$ and the abbreviated symbol $\mathscr{T}(\psi)$ can be utilized.]

So the pairs $(P(\psi_0), |\psi\rangle)$ define on the ensemble of all the quantum mechanical probability chains a *partition* in probability trees. *A fortiori*, they define such a partition also on the ensemble of all the elementary quantum mechanical chain experiments (eqmce) out of which the quantum mechanical random phenomena are made.

Figure 1 provides a simplified example of a probability tree of a state preparation, with only four observables, and making use of somewhat abbreviated notations: *A*, *B*, *C*, *D* are physical observable aspects ("needle positions" of macroscopic devices) corresponding to the quantum mechanical observables A, B, C, D, respectively. The measurement process $M_{C,D}$ corresponds to two commuting observables C, D: (the commutator of C and D is zero, $[C, D]=0$), while M_A and M_B correspond to two noncommuting observables A, B with $[A, B]\neq 0$. The notations $(3')_A$, $(3')_B$, and $(3')_{C,D}$ indicate the observational spaces (3′) corresponding, respectively, to the measurement processes M_A, M_B, and M_{CD} performed on the state represented by $|\psi\rangle = T(t_0, t, H)|\psi_0\rangle$. Each one of the spaces (3′) emerges at some specific time t_A, t_B, and $t_{C,D}$ (a statistical time, defined with respect to the reiterated origin t_0). The commuting observables C, D generate together one common branch producing an observable space (3′) characterized in more detail, namely with respect to both observables involved. $\Delta(P(\psi_0))$, $\Delta(E)$, and $\Delta(P(\psi))$, indicate, respectively, the space-time domains covered by the process of preparation $P(\psi_0)$ of the initial state with state vector $|\psi_0\rangle$, of evolution $E(t_0, t, H)$ represented by $T(t_0, t, H)|\psi_0\rangle$, or, globally, of preparation $P(\psi)=[P(\psi_0), E(t_0, t, H)]$ of the state with state vector $|\psi\rangle$. Here

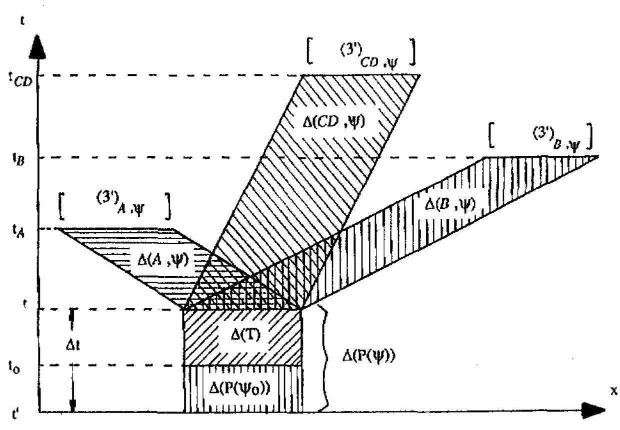

Fig. 1. A quantum mechanical probability tree $\mathscr{T}(P(\psi_0), |\psi\rangle)$.

$\Delta(A, \psi)$, $\Delta(B, \psi)$, and $\Delta(CD, \psi)$, indicate, respectively, the spacetime domains covered by the measurement evolutions M_A, M_B, and M_{CD}.

A quantum mechanical probability tree is a remarkably comprehensive metastructure of probability chains. Most of the fundamental algorithms of the quantum mechanical calculus which combine *one* normed state vector, with the dynamical operators representing the quantum mechanical observables, can be defined *inside—any—one* tree $\mathscr{T}(\mathrm{P}(\psi_0), |\psi\rangle)$:

1. The mean value of an observable A, in a state with state vector $|\psi\rangle$, namely

$$\langle\psi|\mathrm{A}|\psi\rangle, \qquad \forall|\psi\rangle, \quad \forall \mathrm{A}$$

2. The uncertainty theorem, for any pair of observables,

$$\langle\psi|(\Delta\mathrm{A})^2|\psi\rangle\langle\psi|(\Delta\mathrm{B})^2|\psi\rangle \geq |\langle\psi|(i/2)(\mathrm{AB}-\mathrm{BA})|\psi\rangle|$$
$$=(1/2)(h/2\pi), \qquad \forall|\psi\rangle, \quad \forall \mathrm{A}, \mathrm{B}$$

3. The principle of spectral decomposition (expansion postulate)

$$|\psi\rangle = \sum_j c(\psi, a_j)|u_j\rangle$$

$$\forall|\psi\rangle, \quad \forall \mathrm{A}: \quad \mathrm{A}|u_j\rangle = a_j|u_j\rangle \qquad [c(\psi, a_j)\text{: the expansion coefficients}]$$

which permits us to calculate the probability density $\pi(|\psi\rangle, a_j)$ via the probability postulate

$$\pi(\psi, a_j) = |\langle u_j|\psi\rangle|^2 = |c(\psi, a_j)|^2$$

4. Finally, the whole quantum mechanical "transformation theory" from the basis of an observable A to that of an observable B

$$c(\psi, b_k) = \sum_j \alpha_{kj} c(\psi, a_j)$$

$$\forall A, B: \quad \mathrm{A}|u_j\rangle = a_j|u_j\rangle \quad \text{and} \quad \mathrm{B}|v_k\rangle = b_k|v_k\rangle, \quad \forall j \in J, \quad \forall k \in K$$

where J and K are the index sets for the eigenvalues of, respectively, A and B, and $\alpha_{kj} = \langle v_k|u_j\rangle$ are the transformation coefficients.

But as soon as either the principle of superposition or the orthodox quantum mechanical representation of successive measurements comes into play, the corresponding quantum mechanical algorithms cease to be embeddable into one single probability tree: there the embeddability into one tree hits a limit. *Several trees have to be combined.* So a still higher degree of complexity than that of only one probability tree is formed and acts inside the organization implicitly reached by the probabilistic conceptualization hidden inside the quantum mechanical formalism. The quantum mechanical formalism contains implicit calculi with *whole probability trees.*

3. BEYOND KOLMOGOROV'S THEORY OF PROBABILITIES: PROBABILISTIC META AND META-METADEPENDENCE

We have performed an attentive analysis of the connections between Kolmogorov's standard fundamental probabilistic concepts (identically reproducible procedure, universe of elementary events, an algebra of events on this universe, a probability measure on this algebra), the main descriptors of the quantum mechanical formalism (state vectors, operators, eigenfunctions), and the factual counterparts of these. This, *because of the space-time characteristics of the factual counterparts of the quantum mechanical descriptors*, brought forth, with a sort of inner necessity, the probabilistic metaconstruct with treelike space-time support described above. But this metaconstruct of distinct probability chains, though it has been produced by systematic confrontation with the standard probabilistic concepts, *transcends* the abstract theory of probabilities as it now stands: So far the most complex basic probabilistic structure explicitly defined in the theory of probabilities is *one* probability *space*. Not even the notion of *one* probability *chain* is explicitly defined as a monolithic construct. *A fortiori*, the concept of a probability tree, which connects several irreducibly distinct probability chains, is not defined in the current theory of probabilities. Are these novelties probabilistic "anomalies"? Inasmuch as they are rooted in the current abstract theory of probabilities, it seems more adequate to regard them as germs of a possible extension of this theory. Of course, the fact that the quantum mechanical usage of probability measures exceeds the "classical" theory of probabilities was perceived long ago (e.g., Mackey, 1963; Gudder, 1976; Suppes, 1966; Mittelstaedt, 1976; van Fraassen and Hooker, 1976, and many others). But this transgression is usually mentioned in negative terms: "nonembeddability" into a unique probability space of the quantum mechanical measures corresponding to noncommuting observables, which is an "anomaly" that "hinders" a classical definition of a conditional probability for two incompatible events, etc. The concept of probability tree permits us to develop a constructive perception.

3.1. Probabilistic Metadependence Via a Common Potentiality

The quantum mechanical transformation theory ($c(\psi, b_k) = \sum_j \alpha_{kj} c(\psi, a_j)$, $\forall$A, B: $A|u_j\rangle = a_j|u_j\rangle$, $B|v_k\rangle = b_k|v_k\rangle$, $\forall j \in J$, $\forall k \in K$; J, K index sets; A, B two noncommuting observables, $\alpha_{kj} = \langle v_k|u_j\rangle$ the transformation coefficients) permits us to determine entirely, from the knowledge of the probability measure $\pi(\psi, a_j)$ from one branch of a probability tree, any other probability measure $\pi(\psi, b_k)$ belonging to another branch of that same tree. Indeed the equalities $|c(\psi, b_k)|^2 = |\sum_j \alpha_{kj} c(\psi, a_j)|^2$, $\forall j \in J$, $\forall k \in K$, are equivalent to the specification of a funcational relation

$$\pi(\psi, b_k) = F_{QM}[\pi(\psi, a_j)]$$

between the probability measures corresponding to the two noncommuting observables A and B. But the standard concept of functional relation between two probability measures does not singularize the particular sort of probabilistic connection between two probability measures introduced by the quantum theory. *Nor does it permit us to recover it fully*, as shown by Cohen (1988, pp. 991–993). As is stressed by the index QM, we are in the presence of a specifically quantum mechanical functional relation. What status can we assert for it?

According to the current theory of probabilities the concept of "probabilistic dependence" is by definition confined inside *one* probability space where it concerns *isolated* pairs of *events*. Two events are tied by a "probabilistic dependence" if knowledge of one of these events "influences" the expectations concerning the other one. So the relation $\pi(\psi, b_k) = F_{QM}[\pi(\psi, a_j)]$ of mutual determination of the probability measures from a quantum mechanical probability tree can naturally be regarded as a "maximal probabilistic metadependence": "maximal" because it consists in mutual determination; "probabilistic" because, though this determination is a certainty about "influence," nevertheless it concerns probabilistic constructs; "metadependence" because it concerns, not pairs of events from one space, but *globally* pairs of probability measures on entire algebras of events, which, with respect to events, are metaentities. (An explicit definition of this metadependence can be found in Mugur-Schächter (1992c).)

Now, if this view and language are accepted, what has just been named the probabilistic metadependence defined by the quantum mechanical transformation theory appears as *reflecting the studied state with state vector* $|\psi\rangle$ *from the common trunk of the tree.* This state, which stems from a preparation operation $P(\psi_0)$ and then might have evolved according to some law $|T(t_0, t, H)|\psi_0\rangle = |\psi\rangle$, but that has never yet been *observed*, has to be conceived of (in consequence of this lack of previous qualifications) merely as a monolith of still nondifferentiated observational *potentialities* that sets a genetic unity beneath the various incompatible measurement processes of *actualization* of this or that particular observational potentiality, leading to this or that *actualized* observable space (3′). [Though in quite different contexts, Bohm (1951), de Broglie (1956), and Primas (1990), as well as other authors, have also explicitly stressed the multiple potential meaning of the quantum mechanical concept of state.]

The probability tree of a studied state with state vector $|\psi\rangle$ *is a complex unity which, with respect to the observable manifestations of a microsystem, possesses a "potential-actualization-actualized character."*

The quantum mechanical functional relations F_{QM} between the probability measures from irreducibly distinct observable spaces—considered as wholes—belonging to a same probability tree, reflect the genetic unity of these spaces via the common observational potentialities captured inside the state from the trunk of the tree. The quantum mechanical transformation theory involves new probabilistic features that are neither probabilistic "anomalies" nor mere numerical algorithms. They are a mathematical description of particular realizations of probabilistic metaproperties, brought forth by a growth of the probabilistic thinking that happened inside the process of conceptualization of the microphenomena. A growth that draws attention to the necessity, at the most basic level of description where no previously elaborated conceptualization is presupposed, to represent and to study the cognitive *operations* by which the observer—who necessarily exists and acts—*produces* the objects to be qualified and the processes of qualification of these. Indeed, these operations possess *themselves* physical characteristics, in particular space-time supports, that entail nontrivial consequences for the probabilistic descriptions constructed by their help (Mugur-Schächter (1992d)).

3.2. The Germ of a Concept of Probabilistic Meta-Metadependence

3.2.1. State Preparations Versus Measurements

The absence of an integrated perception of the probabilistic organization which underlies the formalism of the quantum theory not only hinders a clear understanding of the novelties and of the problems involved in the theory, but furthermore it entails insufficiencies inside the theory itself. The most important among these stem from the tendency to confound the operations of state preparation with measurements, that is, to mix up temporal orders which, quite essentially, do act. In quantum mechanics as it now stands the degree of definition of the operations of state preparation is much lower than that of the measurement operations. Correlatively, the mutual characterization of operations of state preparation and of measurement operations is very imperfect. The measurement operations are quite explicitly represented by Hermitian linear differential operators and by a well-defined calculus with these. The compatibility or incompatibility of two measurements has been recognized and formally described, and consequences have been drawn systematically from this. On the contrary, a general distinct definition of what is to be called an operation of state preparation, in contradistinction to what has to be called a measurement operation, is uniformly absent. *A fortiori*, the operations of state preparation are not endowed with a mathematical representation clearly assigned to them. They are not even systematically symbolized. *Quantum mechanics as it now stands does not specify a calculus with, specifically, operations of state preparation distinguished from the calculus with measurement operations and related with it.* The term "preparation," nevertheless, is uniformly present.

3.2.2. Superpositions of Several States Versus Spectral Decompositions of One State

The feeble mutual individualization of state preparations and measurements, combined with a fluctuating and feeble distinction between the state vector *of a microsystem* and the eigenvector *of an observable*, entails an insufficient distinction also between the principle of superposition (mainly discussed by Dirac) and Born's principle of spectral decomposition (the expansion postulate). Indeed, though these two principles have been introduced independently of one another, the spectral decompositions of a state vector on the basis of eigenvectors determined by an observable A are currently designated as "superpositions of eigen*states* of A (even by Dirac himself)." The two concepts tend to merge into one another inside the molds of a relaxed language. However:

A *spectral decomposition* $|\psi\rangle = \sum_j c(\psi, a_j)|u_j\rangle$ possesses the following characteristics.

1. It is a representation that is relative, by definition, to some observable A.

2. The *expansion* coefficients $c(\psi, a_j)$ are necessarily *complex* numbers (if they were not, the "interference of probabilities" via transformation to another representation, an essential feature of the formalism, would disappear).

a. They are in general *time dependent* in the Schrödinger representation.

b. The summed *eigen*vectors $|u_j\rangle$ of A, in general an *infinity*, even a *continuous* infinity, are *all* involved, by definition.

c. They are *independent of time.*

d. They are in general *not normalized*, and furthermore not normalizable *strict sensu.*

e. They are *mutually orthogonal by definition*, $\langle u_k|u_j\rangle = 0, \forall(k \neq j)$.

3. Concerning "interference of probabilities":

a. In consequence of the mutual orthogonality of the summed terms, the scalar products $\langle u_j|\psi\rangle$ with individual eigenvectors $|u_j\rangle$ yield one-term results so that for the individual probabilities $\pi(\psi, a_j)$ we have the one-term expressions (7)

$$\pi(\psi, a_j) = |\langle u_j|\psi\rangle|^2 = |c(\psi, a_j)|^2$$

which shows the *ABSENCE of "interference of the probabilities" inside the representation with respect to the ONE observable A itself to which the considered expansion is relative.*

b. While by *passage* to *another* basis corresponding to another observable B ≠ A that does not commute with A, an "interference of probabilities" does appear:

$$\begin{aligned}\pi(\psi, b_k) &= |c(\psi_0, b_k)|^2 \\ &= \left|\sum_j \tau_{kj}(\mathrm{A}, \mathrm{B})c(\psi_0, a_j)\right|^2 \\ &= \sum_j |\tau_{kj}(\mathrm{A}, \mathrm{B})|^2 |c(\psi_0, a_j)|^2 \\ &\quad + [\text{interference terms involving all the pairs of products} \\ &\qquad |\tau_{kj}(\mathrm{A}, \mathrm{B})| c(\psi_0, a_j), |\tau_{jk}(\mathrm{A}, \mathrm{B})| c(\psi_0, a_k)]\end{aligned}$$

This is an *abstract* sort of interference which is *relative to a PAIR of noncommuting observables* (A, B) and which, though it entails certain consequences [as well as many false interpretations (see, e.g., Bohm, 1951, pp. 384–386)] is devoid of a *directly* observable counterpart: the square roots $c(\psi_0, a_j)$ of all the values of the probabilities $\pi(\psi, a_j)$ of the eigenvalues a_j emerging when a measurement of the observable A is performed on a state with state vector $|\psi\rangle$ "interfere" abstractly, numerically, in the value of each probability $\pi(\psi, b_k)$ of an elementary event b_k that *might* emerge *if* a measurement of the observable B that does not commute with A *were* performed on that same state. In what follows this sort of conceived interference by transformation from a representation A to another representation B that never coexists with A will be called *interference relative to incompatible observables.*

On the contrary, *a superposition of states* $|\psi_{abc\ldots}\rangle = \lambda_a|\psi_a\rangle + \lambda_b|\psi_b\rangle + \lambda_c|\psi_c\rangle + \cdots$ possesses the following as if opposed characteristics.

1. It is a representation *not* tied to some particular observable.

2. The coefficients of *linear combination* $\lambda_a, \lambda_b, \lambda_a, \ldots,$ *can* relevantly be chosen to be *real* numbers. Nothing in the formalism interdicts that. They are *time independent.*

3. It is permitted to superpose an *arbitrary* number—usually a *small* number—of state vectors $|\psi_a\rangle, |\psi_b\rangle, \ldots.$

a. They are in general *time dependent* in the Schrödinger representation.

b. They are *always normalized.*

c. *In general they are NOT mutually orthogonal.* However, when in particular they are orthogonal, the scalar products $\langle\psi_a|\psi_{abc\ldots}\rangle$,

$\langle\psi_b|\psi_{abc\ldots}\rangle, \ldots$ acquire one-term expressions $\langle\psi_a|\psi_{abc\ldots}\rangle=\lambda_a$, $\langle\psi_b|\psi_{abc\ldots}\rangle=\lambda_b$, etc., analogous to what happens in the case of a spectral decomposition for the products $\langle u_j|\psi\rangle$. But notice that in this case, in contradistinction to the products $\langle u_j|\psi\rangle=c(\psi_0, a_j)=\sqrt{\pi}(\psi, a_j)$, the values of the "corresponding" products $\langle\psi_k|\psi_{abc\ldots}\rangle=\lambda_k$, $k=a, b, c, \ldots$ do not possess a probabilistic significance.

4. Concerning "interference of probabilities:"

a. The scalar products $\langle u_j|\psi_{abc\ldots}\rangle$ with individual eigenvectors $|u_j\rangle$ from the basis of an observable A do *not* have a one-term expression, they have a multiterm expression $\langle u_j|\psi_{abc\ldots}\rangle=\sum_k \lambda_k\langle u_j|\psi_k\rangle$, $k=a, b, c, \ldots$. So when the square modulus is calculated in order to estimate the corresponding probability $\pi(\psi_{abc\ldots}, a_j)$, an "interference of probabilities" appears *no matter* whether or not the superposed terms $|\psi_a\rangle, |\psi_b\rangle, |\psi_c\rangle, \ldots$ are orthogonal, insofar as these terms are not themselves elements $|u_j\rangle$ from the basis of A (which can happen either in the case of a discrete spectrum or approximately). For instance, for a superposition state vector with only two terms the elementary predictional probabilities concerning the elementary outcomes a_j for an observable A acquire the well-known "interference form"

$$\begin{aligned}\pi(\psi_{ab}, a_j)&=|\langle u_j|\psi_{ab}\rangle|^2\\&=|\lambda_a\langle u_j|\psi_a\rangle+\lambda_b\langle u_j|\psi_b\rangle|^2\\&=|\lambda_a|^2|\langle u_j|\psi_a\rangle|^2+|\lambda_b|^2|\langle u_j|\psi_b\rangle|^2\\&\quad+2\,\mathrm{Re}\{(\lambda_a)(\lambda_b)^*\langle u_j|\psi_a\rangle\langle u_j|\psi_b\rangle^*\}\end{aligned}\tag{8}$$

This is a sort of interference of probabilities where *the quantum mechanical theory of transformation from the basis of one observable to the basis of another observable is not involved*, an interference that emerges "directly" *with respect to the summed state vectors for any ONE observable.* So we shall call it *interference relative to the superposed state vectors* and we shall distinguish it radically from the interference relative to incompatible observables.

b. When one considers, for a superposition state vector, the interference relative to the superposed state vectors that concerns the *position* observable $A=X$, then—if the spatial supports of the superposed state vectors are not disjoint—the corresponding form of the type (8),

$$\begin{aligned}\pi(\psi_{ab}, x_j)&=|\langle\delta(x-x_j)|\psi_{ab}\rangle|^2\\&=|\lambda_a\langle\delta(x-x_j)|\psi_a\rangle+\lambda_b\langle\delta(x-x_j)|\psi_b\rangle|^2\\&=|\lambda_a|^2|\psi_a(x_j)|^2+|\lambda_b|^2|\psi_b(x_j)|^2\\&\quad+2\,\mathrm{Re}\{(\lambda_a)(\lambda_b)^*\psi_a(x_j)\psi_b(x_j)^*\}\end{aligned}\tag{8'}$$

is associated with the amply discussed *interference patterns in the physical space*, directly *observable* on the domains where the spatial supports of the summed state vectors overlap. In this sense the interference relative to the superposed state vectors, in contradistinction to the interference relative to incompatible observables, is not an abstract interference. The possibility of such observable interference patterns disappears only if the spatial supports of the superposed state vectors are all mutually disjoint, in which case (8′) acquires the degenerate noninterferent (but still multiterm) form

$$\pi(\psi_{ab}, x_j) = |\lambda_a|^2 |\psi_a(x_j)|^2 + |\lambda_b|^2 |\psi_b(x_j)|^2 \tag{8''}$$

The mutual specificities emphasized above do not in the least manifest an identity between spectral decompositions of one state vector and superpositions of several state vectors. Quite on the contrary, they manifest a sort of *opposition*. In particular, they reveal *a splitting of the central concept of interference of probabilities*. And whereas the interference relative to incompatible observables has an abstract character, the interference relative to superposed state vectors is tied to directly observable effects.

Now, where do the observable effects tied to superposition state vectors stem from?

In what follows we show that they are essentially related to the "multiple structure" of the operation of state preparation that produces the state corresponding to the considered superposition state vector.

Consider for simplicity a two-term superposition state vector

$$|\psi_{ab}\rangle = \lambda_a |\psi_a\rangle + \lambda_b |\psi_b\rangle \tag{S}$$

tied to a state preparation $P(\psi_{ab})$. If the state $|\psi_b\rangle$ of the studied system is considered *separately*, it stems from some operation of state preparation $P(\psi_a)$. If the state $|\psi_b\rangle$ of the studied system is considered separately, it stems from some operation of state preparation $P(\psi_b)$. If now the superposition state vector $|\psi_{ab}\rangle$ of the studied system is considered, it stems from some operation of state preparation $P(\psi_{ab})$, *again* only "*one*" operation of state preparation since it produces the "one" *pure* state $|\psi_{ab}\rangle$ that entails its own specific quantum mechanical predictions. However, the operation of state preparation $P(\psi_{ab})$ somehow is conceived to "depend" on the two other operations $P(\psi_a)$ and $P(\psi_b)$ that are tied to the two state vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ that *would* have been produced by these operations, respectively, *if* they had been realized *separately*. Implicitly but quite essentially, these other two preparation operations are supposed to be (1) mutually distinguishable, (2) realizable separately, and (3) *combinable* so as to constitute *together* "one" other operation, distinct from both $P(\psi_a)$ and $P(\psi_b)$ and realizable on *one* previous initial state of the studied system, associated with an initial state vector $|\psi_i\rangle$ of that system.

So—quite systematically—in the case of a superposition of state vectors we can write symbolically

$$\mathrm{P}(\psi_{ab}) = f(\mathrm{P}(\psi_a), \mathrm{P}(\psi_b)) \qquad (f\text{: some function})$$

In essence the principle of superposition is a statement, not directly about state vectors, but, more fundamentally, about a—past—operation of state preparation.

But these two different operations of state preparation $\mathrm{P}(\psi_a)$ and $\mathrm{P}(\psi_b)$ involved in the operation $\mathrm{P}(\psi_{ab})$ have *not* been realized separately. They have been realized only "together," "inside" the global procedure $\mathrm{P}(\psi_{ab})$. So the states represented by the corresponding state vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ also, which *could have been* produced separately, individually, via the *separate* realizations of the operations $\mathrm{P}(\psi_a)$ and $\mathrm{P}(\psi_b)$—*which entails that they are normalized*—have *not* been realized separately via $\mathrm{P}(\psi_{ab})$. They are only *conceived of* individually, in relation with the *one* state vector $|\psi_{ab}\rangle$ corresponding to the one realized global operation of state preparation $\mathrm{P}(\psi_{ab})$ (realized either by the observer or by some "natural" substitute of the observer, as in the case of atomic states of an electron); conceived of *AND* explicitly *represented* in the mathematical expression (S) of $|\psi_{ab}\rangle$, where they play a role of *elements of reference* in the calculation of any individual probability $\pi(\psi_{ab}, a_j)$: as can be read from relation (8), $\pi(\psi_{ab}, a_j)$ is a function of $\pi(\psi_a, a_j)$ and $\pi(\psi_b, a_j)$. In particular, when one considers the position observable $\mathrm{A} = \mathrm{X}$ and the corresponding presence probabilities, this reference concerns patterns of impacts observable in the physical space. The algorithm (8) applied to the calculation of an individual presence probability $\pi(\psi_{ab}, x_j)$ as a function of the individual probabilities $\pi(\psi_a, x_j)$ and $\pi(\psi_b, x_j)$ permits, via (8′), *a quantitative comparison* between (1) the observable pattern of position registrations corresponding to the realized state represented by the descriptor $|\psi_{ab}\rangle$ and (2) the patterns that would be produced by each of the states represented by the descriptors $|\psi_a\rangle$ and $|\psi_b\rangle$ if these states acted (or effectively do act) separately on the device for the registration of eigenvalues of the position observable.

What is designated by the term "interference of probabilities" *as applied to observable patterns of position registrations* is precisely the difference brought forth by this comparison between the two patterns corresponding separately to $|\psi_a\rangle$ and $|\psi_b\rangle$ and the pattern corresponding to $|\psi_{ab}\rangle$. One sees how *such patterns are essentially tied to the "multiple" structure* $\mathrm{P}(\psi_{ab}) = f(\mathrm{P}(\psi_a), \mathrm{P}(\psi_b))$ *(f: some function) of the involved operation of state preparation.*

And notice that, remarkably, overlapping of the spatial supports of the superposed state vectors at least somewhere in space-*time* (if time is left to increase indefinitely) is somehow related to a "multiple" structure $P(\psi_{ab}) = f(P(\psi_a), P(\psi_b))$ of the operation of state preparation. Here the fact comes somehow into play (HOW?) that in a superposition of states the combined state vectors are time dependent, while the coefficients of linear combination are not.

We summarize in general terms.

In a superposition representation (S), the *unique* physically realized operation of state preparation is the one symbolized by the notation $P(\psi_{ab\ldots})$, of which the *unique* result is the one symbolized by the global notation $|\psi_{ab\ldots}\rangle$. This—*past*—operation of state preparation $P(\psi_{ab\ldots})$ somehow involved, "contained," two or more other operations of state preparation, $P(\psi_a)$, $P(\psi_b)$, . . . , mutually "distinct" and which *can* be realized *separately*. The state vectors $|\psi_a\rangle$, $|\psi_b\rangle$, . . . , corresponding to the states that *would* have emerged if $P(\psi_a)$, $P(\psi_b)$, . . . , had been accomplished separately, are explicitly specified inside the formal expression of the state vector $|\psi_{ab\ldots}\rangle$ corresponding to the unique physically realized state produced by $P(\psi_{ab\ldots})$. There they play the role of *elements of reference* incorporated into the mathematical representation: It is with respect to them that there emerges a concept of interference of probabilities that is tied to patterns of position registrations directly observable in the physical space.

This is in strong contrast with what is involved in the expression of a spectral decomposition. There the representation does not designate observable effects of a particular type of structure of the past operation of state preparation of the studied state vector. What is represented in a spectral decomposition of a studied state vector $|\psi\rangle$ is the observable effects of a *future* operation of measurement of an observable A performed on $|\psi\rangle$ (Figure 2). The representation is given in terms of the *projections* of the considered state vector $|\psi\rangle$ onto—all—the abstract eigenvectors $|u_j\rangle$, $\forall j \in J$, of the considered observable A. Such an eigenvector $|u_j\rangle$, according to its very definition by the equation for eigenvectors and eigenvalues of A, is not in general a descriptor tied to a state producible by some specific operation of state preparation. It is only part of the mathematical representation of *a framework for the qualification of quantum mechanical states, a framework introduced by the observable* A. Namely, the eigenvectors $|u_j\rangle$, $\forall j \in J$, define a family, specific to this observable A, of "directions of qualification," of "*semantic* directions" (unidimensional, in the absence of degeneracy) each one of which is associated with an observable eigenvalue of A. In general these semantic directions are only tangent to the Hilbert space that contains the state vectors $|\psi\rangle$; they are *exterior* to this space, images of elements endowed with a primary definition only inside the dual of the Hilbert space of the system. By a function (involved in a linear functional on the space of

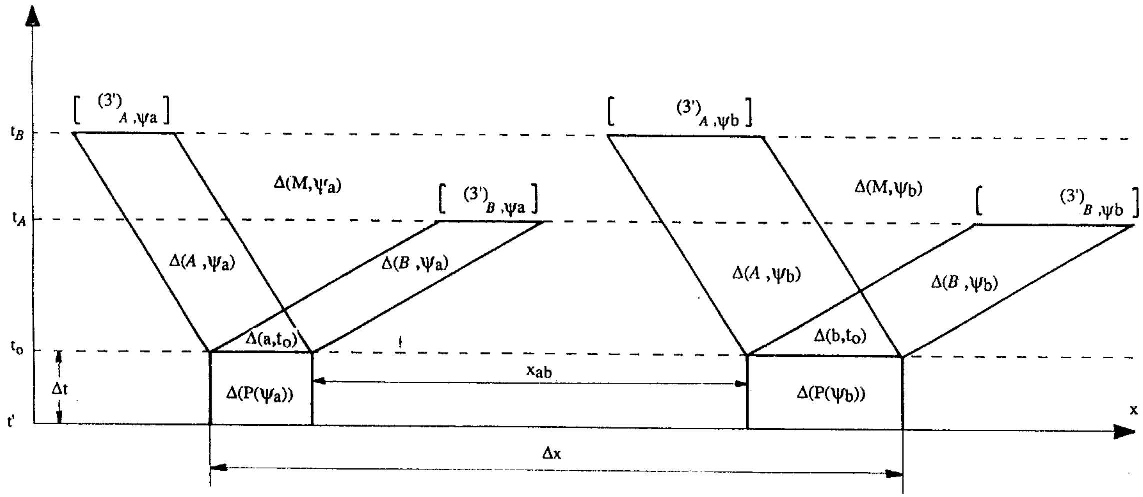

Fig. 2A. The principle of superposition asserted for the state vectors $|\psi_a, t\rangle$ and $|\psi_b, t\rangle$ concerns fundamentally the corresponding operations of state preparation $P(\psi_a)$ and $P(\psi_b)$. These involve processes contained in the space-time domains $\Delta(P(\psi_a))$ and $\Delta(P(\psi_b))$ imbedded in a more extended space-time domain $\Delta x\, \Delta t$, where the temporal extension $\Delta t = t \approx t'$ is limited by the time t when the studied state vectors $|\psi_a, t\rangle$ and $|\psi_b, t\rangle$ emerge. While the principle of spectral decomposition concerns measurement operations involving space-time domains $\Delta(A, \psi_a)$, $\Delta(B, \psi_a)$, etc., imbedded in a more extended space-time domain $\Delta(M, \psi_a)$, or space-time domains $\Delta(A, \psi_b)$, $\Delta(B, \psi_b)$, etc., imbedded in a more extended space-time domain $\Delta(M, \psi_b)$, the temporal extensions of $\Delta(M, \psi_a)$ and $\Delta(M, \psi_b)$ being posterior to the time t. The spatial distance x_{ab} that separates the domains $\Delta(a, t)$ and $\Delta(b, t)$, where the state vectors $|\psi_a, t\rangle$ and $|\psi_b, t\rangle$, respectively, assert a nonnegligible presence probability, is arbitrarily big. As in Figure 1, the notations $[3'_A, \psi_a]$, $[3'_B, \psi_a]$, $[3'_A, \psi_b]$, and $[3'_B, \psi_b]$ symbolize probability spaces brought forth by the considered measurement processes.

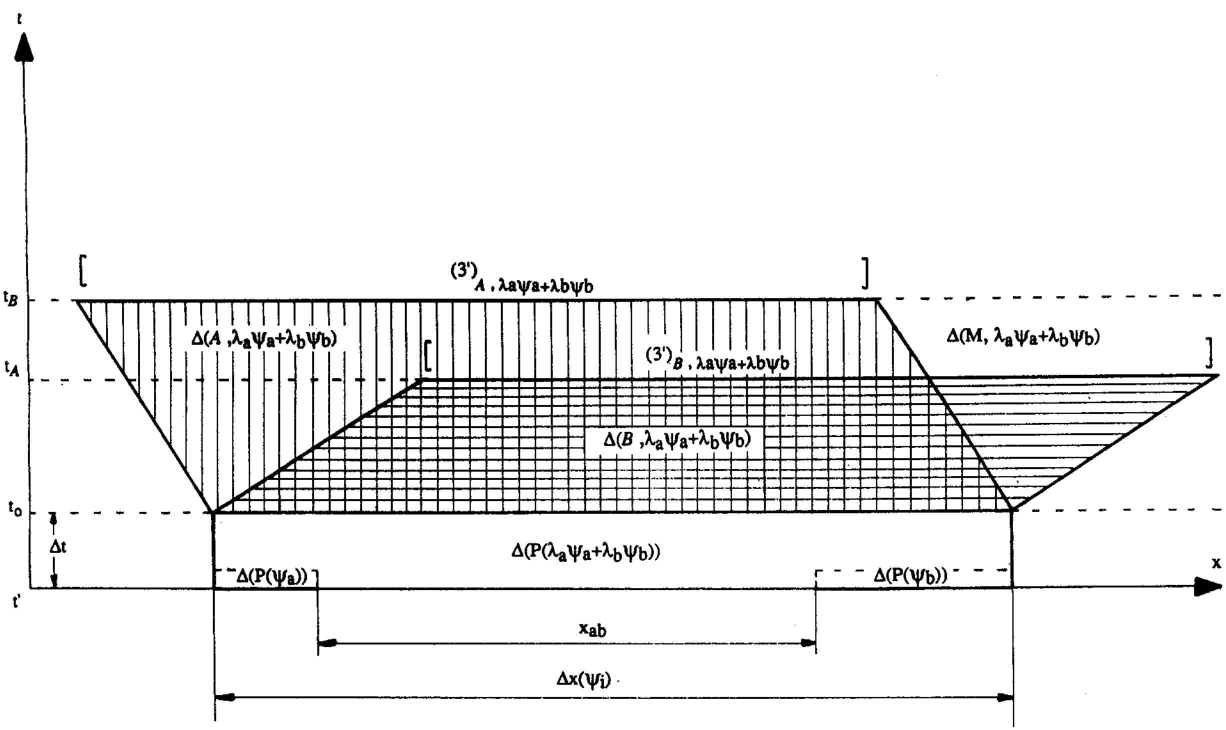

Fig. 2B. According to the principle of superposition asserted concerning the state vectors $|\psi_a, t\rangle$ and $|\psi_b, t\rangle$, no matter how big is the distance x_{ab} defined in Fig. 2A, the separately realizable operations of state preparation $P(\psi_a)$ and $P(\psi_b)$ can be combined in a unique operation $P(\psi_{ab})$ acting on an initial state $|\psi_i, t'\rangle$, $t' < t$, of the studied system that possesses a corresponding extension $\Delta x(\psi_i)$. The unique operation $P(\psi_{ab})$ which out of $|\psi_i\rangle$ brings forth the superposition $|\psi_{ab}, t\rangle = \lambda_a|\psi_a, t'\rangle + \lambda_b|\psi_b, t\rangle$ develops inside a space-time domain $\Delta(P(\lambda_a\psi_a + \lambda_b\psi_b))$ imbedded in the more extended domain $\Delta x(\psi_i)\,\Delta t$, where $\Delta t = t - t'$. Measurement operations of two noncommuting observables A and B performed on the superposition state vector $|\psi_{ab}, t\rangle = \lambda_a|\psi_a, t\rangle + \lambda_b|\psi_b, t\rangle$ develop, respectively, inside space-time domains $\Delta(A, \lambda_a\psi_a + \lambda_b\psi_b)$ and $\Delta(B, \lambda_a\psi_a + \lambda_b\psi_b)$ of which the temporal extensions $t_A - t$ and $t_B - t$ are posterior to the time t. These measurement operations bring forth probability spaces $[3'_A, \lambda_a\psi_a + \lambda_b\psi_b]$, $[3'_B, \lambda_a\psi_a + \lambda_b\psi_b]$.

the states) the eigenvector $|u_j\rangle$ corresponding to an eigenvalue a_j of an observable A *qualifies* some feature (which one exactly?) of the same global factual situation that is also qualified by the eigenvalue a_j. As to the eigenvalue a_j itself, it qualifies the individual observable outcomes V_j, with $f_A(V_j) = a_j$ of the elementary quantum mechanical chain experiments, which, in their turn, via the corresponding probabilistic metaqualifications $\pi(\psi, a_j)$, qualify globally what is called a "quantum mechanical state" and is represented by a state vector $|\psi\rangle$. We sum up:

A spectral decomposition $|\psi\rangle = \sum_j c(\psi, a_j)|u_j\rangle$ is referred to a *future* operation of measurement of an observable A, upon the studied—already prepared—state vector $|\psi\rangle$. Each eigenvector $|u_j\rangle$ of A is a descriptor of a particular *qualification* from a whole *framework for qualification* introduced by A, a framework that is defined on the whole space of the state vectors.

Though a descriptor $|u_j\rangle$ is utilized for calculating the probability of an outcome $f_A(V_j) = a_j$ for any given state vector, there is *nothing* probabilistic in this descriptor *itself*. The descriptor $|u_j\rangle$ is tied to *one* eigenvalue a_j (in a nondegenerate situation), so it points toward an essentially individual predication. There is *no reason* whatever to require *normability* for the mathematical descriptor $|u_j\rangle$, as for the state vectors $|\psi\rangle$, which—by definition—generate probability measures. Quite on the contrary, this would simply be grossly inadequate from a semantic point of view.

Correlatively, the spectral decomposition with respect to one observable A—by itself—entails no interference of probabilities, neither observable nor abstract. An (abstract) interference of probabilities tied to spectral decompositions arises only by transformation from the basis of one observable A to the basis of another observable B that does not commute with A.

Since the eigenvectors are descriptors with individual meaning, the "problem" of normalization of the eigenvectors of observables with continuous spectrum is a false problem. So the "resolution" by the construction of state vectors yielding approximated normed representations of eigenvectors is a resolution without a corresponding problem, just noxious, mathematically generated semantic fog that masks under a veil of superficial uniformity a radical solution of continuity, in the space of the concepts, between eigenvectors and state vectors. Even the standard theory of probabilities rejects (implicitly of course) the confusion between eigenvectors and state vectors. This, for instance, is illustrated by very interesting remarks by Cohen [Cohen, 1988, pp. 991–992, equations (54)–(59)]. In order to understand deeply the veritable problem involved in the quantum mechanical description of measurements, in order to formulate it in better analyzed terms and to form a veritable answer to it, the conceptual difference between the designata of the eigenvectors and those of state vectors has to be recognized as essential, to be specified, and to be set at the bottom.

In short, the code is in essence as follows for distinguishing between the factual counterparts of, on the one hand, the superposition writings, and on the other hand, the spectral decompositions:

1. A linear combination of an *arbitrary* number of (in general) *time-dependent* and mutually *non*orthogonal "*state*" vectors *of a system S*, all *normalized*, that is not relative to some observable and that can, in particular, be relevantly written with *real* coefficients, can be regarded as: The formal expression of the result of *one* operation of state preparation somehow "depending" on (referrable to) other (two or more) operations of state preparation, individually realizable but not individually realized, and which are such that if they were individually realized, would produce the states corresponding to the linearly combined state vectors.

2. A linear expansion of *one normalized* "*state*" vector, on the basis of *all* the mutually *orthogonal* and (in general) infinitely numerous and *nonnormalizable* "*eigen*"vectors *of an observable A*, with *complex* and *time-dependent* expansion coefficients, can be considered as: A formal expression of the qualification of the physical state represented by that state vector, inside the framework for qualification of *any* quantum mechanical state introduced by the observable A; namely, a probabilistic qualification of the state by the probability densities $|c(\psi, a_j)|^2 = |\langle u_j|\psi\rangle|^2$ of the observable outcomes $f_{\mathrm{A}}(V_j) = a_j$ of the quantum mechanical elementary chain experiments performed with that state and with the measurement evolutions M_A for A.

In particular, it can happen that the spectrum of the considered observable A is intrinsically discrete (Hamiltonian of a bounded state or a kinetic momentum). This entails then an identification of each eigenvector with a state vector of a preparable state [which involves then also a definite finite norm for the eigenvectors, as well as mutual orthogonality, and independence of time for the ensemble (a discrete infinity) of these "eigen-state vectors"]. Nevertheless, even in these particular situations which introduce for each eigenstate vector a cumulation of two distinct roles, the conceptual difference still quite fully subsists between the designatum of a superposition of several eigenstate vectors on the one hand (way of preparing the superposition state vector), and on the other hand the designatum of a decomposition of a state vector along the whole infinity of eigenstate vectors from the basis of eigenvectors of the considered observable (way of qualifying that state vector). And the existence of this difference continues to be even formally disclosed by the subsistence of the possible relevance, or not, of real coefficients.

So the code explicated above always avoids confusion between superpositions and spectral decompositions. (The removal of this confusion might lead to a clear understanding of the superselection rules. It might also clarify the significance of conceptually rather obscure perturbation methods used for the calculation of the spectrum of the energy of quantified systems, etc.) But resort to the code ceases to be necessary as soon as one is in possession of the concept of probability tree. By the simple contemplation of the figure that represents a tree, it becomes *obvious* that the superpositions concern the *primary* operation $P(\psi_0)$ of generation of an *object* for subsequent examinations, while the decompositions concern the *secondary* operations M_A of *qualification* of this object (Figure 2). It jumps to the eye that these two concepts concern essentially *different phases* of the genesis of the quantum mechanical events, placed at two different temporal levels of a tree, imbedded in different space-time domains and *possessing essentially different cognitive roles.*

3.2.3. The Germ of a Calculus with Whole Probability Trees (Probabilistic Meta-Metadependence)

The quasiconfounded treatment of superpositions and of spectral decompositions hides the important fact that, in a certain sense, a superposition of states—but not also a spectral decomposition—involves a germ of *a calculus with several probability trees, globally considered.*

Consider a state vector $|\psi\rangle$ which is instructionally defined by the specification of only one preparation procedure $P(\psi)$. Then the probability measures from the observational spaces (3′) of the corresponding probability tree are completely specified by reference to the only *one* state vector $|\psi\rangle$ tied to the unique operation of state preparation $P(\psi)$ (for simplicity we suppose measurements directly on the prepared state $|\psi\rangle$, i.e., we consider the particular case $t-t_0=0$, $|\psi\rangle\equiv|\psi_0\rangle$). For example, in $(3')_A$ the measure $\pi(\psi, a_j)$ is calculable on the basis of the postulate (2), $\pi(\psi, a_j)=|\langle u_j|\psi\rangle|^2$, by making use exclusively of the state vector $|\psi\rangle$. But the situation changes if we consider a superposition state vector

$$|\psi_{ab}\rangle=\lambda_a|\psi_a\rangle+\lambda_b|\psi_b\rangle$$

(as, for example, in the case of Young interference). Then—physically—the corresponding preparation $P(\psi_{ab})$ still introduces only one state $|\psi_{ab}\rangle$, so only one probability tree. Nevertheless, as has been stressed, the probability measures from the observable spaces (3′) of this unique tree are now calculated by reference to, also, the two state vectors $|\psi_a\rangle$ and $|\psi_b\rangle$ from the mathematical expression of $|\psi_{ab}\rangle$. This happens algorithmically, via the combination of (a) the additive quantum mechanical representation of the state $|\psi_{ab}\rangle$ by a superposition writing, (b) the spectral decomposition writings, and (c) the probability postulate (2). Indeed, when according to (S) and (2) the measure $\pi(\psi_{ab}, a_j)$ has to be calculated by the use of the relation (8),

$$\pi(\psi_{ab}, a_j) = |\lambda_a|^2 |\langle u_j|\psi_a\rangle|^2 + |\lambda_b|^2 |\langle u_j|\psi_b\rangle|^2 + 2\,\mathrm{Re}\{(\lambda_a)(\lambda_b)^* \langle u_j|\psi_a\rangle \langle u_j|\psi_b\rangle^*\}$$

or, in particular, (8′),

$$\pi(\psi_{ab}, x_j) = |\lambda_a|^2 |\psi_a(x_j)|^2 + |\lambda_b|^2 |\psi_b(x_j)|^2 + 2\,\mathrm{Re}\{(\lambda_a)(\lambda_b)^* \psi_a(x_j) \psi_b(x_j)^*\}$$

then *three* probability trees are brought into play—globally—namely the unique tree $\mathcal{T}(\psi_{ab})$ physically generated by the unique physically realized preparation $P(\psi_{ab})$ and the two trees $\mathcal{T}(\psi_a)$ and $\mathcal{T}(\psi_b)$ corresponding to the two preparations $P(\psi_a)$ and $P(\psi_b)$ on which the preparation $P(\psi_{ab}) = f(P(\psi_a), P(\psi_b))$ "depends"—considered separately—which have not been realized individually, but, being reflected in the writings by the specification of their possible individual results $|\psi_a\rangle$ and $|\psi_b\rangle$, act there as a conceptual reference. In fact, what is brought into play is *a structure of three mutually consistent rules of "formal composition,"* namely the rule of composition of:

1. The reference-preparation operation $P(\psi_a)$ with the reference-preparation operation $P(\psi_b)$: *Some* definition *of the function* $f(P(\psi_a), P(\psi_b) = P(\psi_{ab}))$ *and of its physical counterpart are supposed to "exist": this supposition in fact* constitutes *the essence of the principle of superposition. However, this basic definition is not spelled out inside quantum mechanics as it now stands.*

2. The reference-state vector $|\psi_a\rangle$ corresponding to the preparation operation $P(\psi_a)$, with the reference state vector $|\psi_b\rangle$ corresponding to the preparation operation $P(\psi_b)$ [the additive rule (S)].

3. The corresponding reference-observable probability measure $|\lambda_a\langle u_j|\psi_a\rangle|^2$, with the reference observable probability measure $|\lambda_b\langle u_j|\psi_b\rangle|^2$ [the quantum mechanical algorithm $(2+\mathrm{S}) = (8)$].

Globally, what comes here in implicitly is a complex algorithm of formal composition of the two only conceived reference probability trees $\mathcal{T}(\psi_a)$ and $\mathcal{T}(\psi_b)$, such as to yield, *by a sort of "probabilistic dependence" defined between entire trees*, precisely the result postulated by the relation (2) for the probability measures from the unique tree $\mathcal{T}(\psi_{ab})$ which is physically realized. *Such an algorithm amounts to endowing the mathematical representation assigned to* each level *of the unique physically realized tree (operation of state preparation, prepared state vector, observable probability space), with an incorporated* reference *to the corresponding level of the two other, only conceived trees.*

Obviously, such a representation, endowed with such a reference, transgresses essentially the concept of one probability tree; it involves certain metaqualifications with respect to the qualifications which can be expressed inside the nonreferred representation of one single tree. We are here in the presence of a probabilistic meta-metadependence with respect to the present standard concept of probabilistic dependence (since the quantum theory of transformations involves already—inside a unique tree—a sort of probabilistic metadependence with respect to the probabilistic dependence in the sense of the theory of probabilities as it now stands). Only if this probabilistic meta-metadependence, globally considered, is taken into consideration also does it become possible to try to encompass the whole significance of the quantum mechanical principle of superposition.

Thus, inside quantum mechanics as it now stands, the germ of certain algorithms can be discerned corresponding to an implicit calculus with entire probability trees. This happens each time that superposition states are represented. (This happens also each time that successive measurements are represented. But then the conceptual insertion is different: Instead of the principle of superposition, the projection postulate acts at the bottom, identifying the operations of preparation in the general sense, with the particular category of preparations by measurement evolutions. This distorts the topology and flattens the volume of the conceptual space involved.) However, with the implicit and incomplete quantum mechanical calculus with entire probability trees we penetrate into this confused frontier zone—which always does exist—where the representations already elaborated by a theory plunge into the still unconceptualized.

The basic lacuna is that the operations of state preparation are devoid of mathematical representation.

This is a lacuna of which the consequences mark the intelligibility of the whole orthodox formalism. Below we compensate for it.

4. OPERATORS OF STATE PREPARATION AND THE PRINCIPLE OF SUPERPOSITION

4.1. Operators of State Preparation and Their Calculus

What operators and what calculus with these can be defined in order to represent mathematically the physical operations of state preparation in a way that is consistent with the orthodox formalism as it now stands?

Suppose that $G(\psi)$ (G: generator) is an operator that represents mathematically the operation of state preparation $P(\psi)$. For consistency with the linear formalism of quantum mechanics let us require $G(\psi)$ to be a *linear* operator. Then, to represent mathematically the preparation of the states with state vectors $|\psi_a\rangle$, $|\psi_b\rangle$, and $|\psi_{ab}\rangle = \lambda_a|\psi_a\rangle + \lambda_b|\psi_b\rangle$, we have to write, respectively, for any choice of some initial state vector $|\psi_i\rangle$,

$$G(\psi_a)|\psi_i\rangle = |\psi_a\rangle, \qquad G(\psi_b)|\psi_i\rangle = |\psi_b\rangle, \qquad G(\psi_{ab})|\psi_i\rangle = |\psi_{ab}\rangle$$

[read: $G(\psi_a)$ acting on some—any—previously existing state with state vector $|\psi_i\rangle$ (known or *not*) generates out of it the state with state vector $|\psi_a\rangle$, etc.]. The unknown functional relation f from the representation $P(\psi_{ab}) = f(P(\psi_a), P(\psi_b))$ concerning the three *factual* operations $P(\psi_{ab})$, $P(\psi_a)$, and $P(\psi_b)$ involved in the preparation of a superposition state vector $|\psi_{ab}\rangle = \lambda_a|\psi_a\rangle + \lambda_b|\psi_b\rangle$ will somehow translate into a formal relation g, $G(\psi_{ab}) = g[G(\psi_a), G(\psi_b)]$. To find the translation, we write down the conditions, in agreement with the linearity required for the $G(\psi)$,

$$\begin{aligned} G(\psi_{ab})|\psi_i\rangle = |\psi_{ab}\rangle = \lambda_a|\psi_a\rangle + \lambda_b|\psi_b\rangle &= \lambda_a G(\psi_a)|\psi_i\rangle + \lambda_b G(\psi_b)|\psi_i\rangle \\ &= [\lambda_a G(\psi_a) + \lambda_b G(\psi_b)]|\psi_i\rangle \end{aligned} \tag{9}$$

The function g that connects the operators $G(\psi_{ab})$, $G(\psi_a)$, and $G(\psi_b)$ is the same linear combination that connects the state vectors $|\psi_{ab}\rangle$, $|\psi_a\rangle$, and $|\psi_b\rangle$. So in general terms

$$\begin{aligned} g[G(\psi_a), G(\psi_b), G(\psi_c), \ldots] &= G(\psi_{abc\ldots}) \\ &= \sum_k \lambda_k G(\psi_k), \qquad k = a, b, c, \ldots \end{aligned} \tag{10}$$

Furthermore, since for the well-known quantum mechanical operator of projection onto $|\psi\rangle$, P_ψ, we have $P_\psi|\psi_i\rangle = |\psi\rangle\langle\psi|\psi_i\rangle$, $\forall|\psi_i\rangle$, while by definition $G(\psi)|\psi_i\rangle = |\psi\rangle$, we can write

$$G(\psi) = (1/\langle\psi|\psi_i\rangle)P_\psi \tag{11}$$

which we shall call a "normalized projector" onto $|\psi\rangle$:

A "normalized" projector P_ψ yields an adequate representation for the concept of an "operator $G(\psi)$ of state preparation" such as it is required by (10).

From (10) and (11) it follows that for a superposition state vector $|\psi_{abc\ldots}\rangle$ we can write

$$\begin{aligned} G(\psi_{abc\ldots}) &= (1/\langle\psi_{abc\ldots}|\psi_i\rangle)P_{\psi_{abc\ldots}} \\ &= \left(\sum_k \lambda_k \Big/ \langle\psi_k|\psi_i\rangle\right)P_{\psi_k}, \qquad k = a, b, c, \ldots \end{aligned} \tag{12}$$

The operator of preparation of a superposition state vector can be represented mathematically by a linear combination of normalized projectors.

Which includes automatically the *particular* case of preparation by a measurement evolution M_A posited by the orthodox projection postulate: In that case the state preparation operator becomes indeed $(1/\langle u_j|\psi_i\rangle)P_{|u_j\rangle}$, where $|u_j\rangle$ is the eigenvector of the observable A corresponding to the registered eigenvalue a_j. But it has to be clearly realized that in the formalism as it now stands the projectors P_ψ are *not* utilized with the fundamental role of general formal representatives of the operations of state preparation. The projectors P_ψ are utilized currently in the algorithms connected with *measurement* operations [density (or statistical) operators].

The definition (11)+(12) has interesting implications concerning the coherence between the semantics to be assigned to the formal feature of commutativity of two linear operators and the nonformalized qualification of "compatibility" drawn from the current language:

For consistency with the linear formalism of quantum mechanics we have required linearity for the operators of state preparation. This entailed the necessity, in (12), of a linear superposition of *distinct*, so *non*commuting normed projectors P_{ψ_a}, $P_{\psi_b}, \ldots$ that shall *all* act on *one* same initial state vector $|\psi_i\rangle$, while two commuting projectors—which reduce in fact to one single projector—cannot generate a superposition state vector because they (it) cannot represent the required *distinct* actions on *one* same initial state vector $|\psi_i\rangle$. In *this* sense:

For the mathematical representation of the process of generation of a superposition state, distinct and non*commuting operators of state preparation are* "compatible" *operators.*

This is as if "opposite" to what happens for the mathematical representation of the operations of measurement of dynamical quantities: two *dynamical* operators, as is well known, are considered to be "compatible" when they commute, while if they do not commute they are considered to be "incompatible."

Now, we have emphasized that in the case of the representation of measurement operations, the factual counterpart of the "compatibility" of two—commuting—dynamical operators A and B consists of the possibility of *individual* measurement evolutions M_{AB} for A and B possessing one *common* space-time support. This is what entails the possibility, from each (one) registered "needle position" V_j that has been the unique factual observable outcome of one given reiteration of a measurement evolution M_{AB}, to calculate a *pair* of two correlated eigenvalues $a_j = f_A(V_j)$, $b_j = f_B(V_j)$ (which is verbally designated as the possibility of a "simultaneous" measurement of the observables A and B). While if A and B do not commute, the individual measurement evolutions M_A for A and M_B for B possess necessarily distinct space-time supports, which is designated by the assertion that "simultaneous measurements for A and B are not possible" (the factual substance of Bohr complementarity).

In short: When exclusively measurement operators are considered, the two qualifications "commuting" and "compatible" apply to the same subensemble of operators, so that they tend to be identified. But when also normalized projectors as representatives of operations of state preparation are considered, the domains of application of these two qualifications separate. So a new language emerges which concerns a more complex situation. We shall now establish explicitly this new language. Take into account:

1. The usage of language found above and the corresponding designata for the case of measurements.

2. The fact that two different projectors do not commute, while two commuting projectors become identified.

3. The fact that the different projectors involved in the preparation of a superposition state represent individual operations that are physically different and, nevertheless, can all act on one same initial—individual—factual situation corresponding to one same initial quantum mechanical state vector $|\psi_i\rangle$.

4. The systematic distinction between abstract descriptor and its physical designatum.

5. The systematic distinction between (a) the *individual* level of description (where are placed the various individual realizations of an operation of state preparation, or of a measurement evolution, or of an elementary chain experiment), and (b) the metalevel of *probabilistic* description (where is placed by definition the quantum mechanical concept of state vector $|\psi\rangle$) and, correlatively, the concept of "one" (complete) quantum mechanical measurement involving a whole ensemble of elementary chain experiments.

6. The requirement of one same stable language valid no matter whether measurements or state preparations are described.

The elements listed above entail together the following rather complex dictionary.

1. "Compatibility" or "noncompatibility" of two linear operators (dynamical or not): respectively, the relevance or not of the action of both these operators on *one INDIVIDUAL* realization of a state of the studied system corresponding to one given quantum mechanical state vector.

2. "Commutativity" or "noncommutativity" of two linear operators (dynamical or not): respectively, the identity or the disjoint character of the space-*time* supports of the individual physical operations represented by these two operators.

3. *Multiplicative* composition of the action of two (or more) commuting *dynamical* operators upon one given state vector $|\psi\rangle$: mathematical expression of the *factual* identity of two (or more) processes of *qualification* of any

one individual realization of a state of a system corresponding to $|\psi\rangle$, via *one* common sort of *individual* measurement evolutions $M_{AB\ldots}$ realizable on one same space-time support, but of which the—one, common—*factual* observable outcome V_j, once it has emerged, can be then *conceptually* worked out in *various* ways, $a_j = f_A(V_j)$, $b_j = f_B(V_j)$, etc. (which justifies the above somewhat misleading wording "two or more" processes of qualification).

4. *Additive* composition of two (or more)—necessarily—noncommuting operators *of state preparation* (normalized projectors) upon one given initial state vector $|\psi_i\rangle$: mathematical expression of the generation, out of any *one* realization of an individual factual state of the studied system tied to the quantum mechanical state vector $|\psi_i\rangle$, of one realization of a new factual state of the studied system tied to a new quantum mechanical state vector $|\psi_0\rangle$ via the action of two (or more) *factually* different processes of "preparation" possessing disjoint space-time supports, all these processes being posited to end at a same moment, which is the initial moment t_0 of the newly prepared state vector $|\psi_0\rangle = |\psi(t_0)\rangle$.

With this dictionary, we can now say that:

In the case of the representation of an operation of state preparation

$$\begin{aligned} G(\psi_{abc\ldots}) &= (1/\langle \psi_{abc\ldots}|\psi_i\rangle) P_{\psi_{abc\ldots}} \\ &= \sum_k (\lambda_k/\langle \psi_k|\psi_i\rangle) P_{\psi_k}, \qquad k = a, b, c, \ldots \end{aligned}$$

that generates a superposition state $|\psi_{abc\ldots}\rangle = \sum_k (\lambda_k|\psi_k\rangle$, the distinct *noncommuting* normalized projectors $(1/\langle \psi_{abc\ldots}|\psi_i\rangle) P_{\psi_k}$ that are involved correspond to *compatible* physical actions of which nevertheless the space-time supports are *disjoint*.

So quantum mechanics permits (could we even say that it requires?) a certain *coherent prolongation* of its formalism and its language, where the operations of state preparation (all of them, not only those consisting of measurement evolutions M_A) are mathematically represented by operators of state preparation $G(\psi)$ that are normalized projectors combined according to a specific calculus entailed by the fact that the space of the normalized kets $|\psi\rangle$ is posited to be a vector space. This calculus with operators of state preparation is distinct from the calculus with dynamical operators, which represent measurements and are tied to the principle of spectral decomposition. This finally *demonstrates* that the formal structure of the quantum theory by no means entails the orthodox flattening identifications between preparations and measurements and between superpositions of several state vectors and spectral decompositions of one state vector: It rejects them in fact, if we go to the bottom.

The definition (11) + (12) of operators of state preparation effaces the lacuna in the rules of combination of two or more probability trees regarded as wholes. So the implicit quantum mechanical calculus with whole probability trees, expressing a new probabilistic concept of probabilistic meta-meta-dependence, is now entirely explicated. But the most important consequence is the elucidation of the physical significance of the principle of superposition, stated below.

4.2. The Minimal Model Involved by the Principle of Superposition

In quantum mechanics as it now stands the mathematical expression of the principle of superposition is referred exclusively to the state vectors. This is misleading. Indeed—fundamentally—the principle of superposition refers to operations of state preparation. And the definition (11) + (12) is equivalent to a deepened reformulation involving now explicitly these operational roots also. This permits progress concerning the physical implications of the principle.

Consider a two-term superposition state vector $|\psi_{ab}\rangle = \lambda_a|\psi_a\rangle + \lambda_b|\psi_b\rangle$. We have shown that in order to represent mathematically the operation of preparation of $|\psi_{ab}\rangle$ we must make use of a normalized projector

$$(1/\langle\psi_{ab}|\psi_i\rangle)\mathrm{P}_{\psi_{ab}} = (\lambda_a/\langle\psi_a\psi_i\rangle)\mathrm{P}_{\psi_a} + \lambda_b/\langle\psi_b|\psi_i\rangle\mathrm{P}_{\psi_b}$$

that is a linear combination of two distinct normalized projectors $(1/\langle\psi_a\psi_i\rangle)\mathrm{P}_{\psi_a}$ and $(1/\langle\psi_b|\psi_i\rangle)\mathrm{P}_{\psi_b}$ which act on *one* initial state vector $|\psi_i\rangle$ out of which they generate $|\psi_{ab}\rangle$:

$$[(\lambda_a/\langle\psi_a|\psi_i\rangle)\mathrm{P}_{\psi_a} + \lambda_b/\langle\psi_b|\psi_i\rangle\mathrm{P}_{\psi_b}]|\psi_i\rangle = |\psi_{ab}\rangle$$

We have also shown that this mathematical representation involves the assumption of "compatibility" of the physical processes described by the two operators $(1/\langle\psi_a\psi_i\rangle)\mathrm{P}_{\psi_a}$ and $(1/\langle\psi_b|\psi_i\rangle)\mathrm{P}_{\psi_b}$, in a definite sense which concerns the space-time features of the mentioned processes. Now, in consequence of the conditions of norm, the two spatial domains $\Delta(|\psi_a|^2, t) = \Delta(a)$ and $\Delta(|\psi_b|^2, t) = \Delta(b)$, where $|\psi_a\rangle$ and $|\psi_b\rangle$, respectively, yield presence probabilities that are not quasinull, are *finite* with respect to any fixed definition of quasinullity. And, since the current formulation of the principle of superposition asserts that the state represented by $|\psi_{ab}\rangle$ can be created for *any* pair $|\psi_a\rangle$ and $|\psi_b\rangle$, we are free to imagine in particular that $|\psi_a(x, t)\rangle$ and $|\psi_b(x, t)\rangle$ are such that, at a given time t (in the observer's referential) the two spatial domains $\Delta(a, t)$ and $\Delta(b, t)$ are disjoint *and* the (purely spatial) distance that separates them is very big, say, of the order of light-years. Nevertheless, as it is explicitly expressed by the new expression of the principle of superposition

$$[(\lambda_a/\langle\psi_a|\psi_i\rangle)P_{\psi_a}+\lambda_b/\langle\psi_b|\psi_i\rangle P_{\psi_b}]|\psi_i\rangle=|\psi_{ab}\rangle$$

quantum mechanics still assumes that there *does* exist an initial state vector $|\psi_i(x, t')\rangle$, $t'<t$, of the *ONE* considered "system," such that the two preparation processes represented by the two mathematical writings $(1/\langle\psi_a\psi_i\rangle)P_{\psi_a}$ and $(1/\langle\psi_b|\psi_i\rangle)P_{\psi_b}$ can both take place "compatibly" on each individual realization of a factual state corresponding to the state vector $|\psi_i(x, t')\rangle$. But this is a model (Figures 1 and 2):

The principle of superposition associates to the entity called "one" quantum system a *model* according to which an individual factual realization of a state of this entity *can* be such that—whatever be other nonspecified qualifications of it—this state *covers an arbitrarily big spatial domain*, notwithstanding that in some (nonspecified) sense a quantum system is conceived to be also "microscopic" (it is even often called "one microsystem").

Horrible dictu, but the orthodox formulation, though it proclaims interdiction of any model, in fact is itself *founded* on a model. And this model, while on the one hand it violates the natural slopes of the connection between what we would agree to call a microsystem and the designatum forced upon us by the principle of superposition, on the other hand is not achieved, not worked out. In this sense it is a "minimal" model. Camouflaged loosely inside the conceptual volume delimited by its noncommittal absence of full specification, this minimal model fluctuates there implicitly leading to confusion and perplexity. Whether it is explicitly declared or not, this minimal model *is* there, encapsulated into the principle of superposition. And it acts on our speaking and on our thinking. It literally invades them in the form of problems and paradoxes that haunt the quantum theory ever since it appeared. "Schrödinger's cat" or more abstractly "the reduction problem," as well as the "locality" problem, are only the most striking distillations and scandalous amplifications of consequences of this hidden unfinished model. Only further specifications could remove the ambiguities that emanate from this model, and perhaps thereby also its queerness.

5. A MEASUREMENT THEOREM

We have brought forth a radical distinction between, on the one hand, preparations and superpositions of several state vectors, and on the other hand, measurements and spectral decompositions of one state vector. We shall now try to understand more clearly how these two distinct pairs of concepts are *related*.

Bohm (1951), de Broglie (1956), Park and Margenau (1968) (in their study of the "time of flight" method for the measurement of the momentum observable), as well as other authors, have already strongly and variously emphasized that an evolution law of the descriptor $|\psi(x, t)\rangle$, if it is "good" for producing "measurement evolutions" M_A of the first kind for an observable A, possesses specific characteristics. Nevertheless, quantum mechanics as it now stands does not introduce an explicit general definition of the operator of evolution H_A to be tied with the individual measurement evolutions M_A corresponding to a dynamical observable A. It only supposes implicitly that, given a "physically significant" quantum mechanical observable A (as is well known, not any quantum observable is measurable), such an operator H_A *can* be found for A. Below we introduce a condition that ensures some of the characteristics identified by other authors.

Condition CH_A. A quantum mechanical evolution operator H_A can be connected with the *individual* measurement evolutions M_A of the first kind corresponding to a quantum mechanical observable A only if it works like an operator

$$[1/\langle\chi(x, t')|\psi(x, t)\rangle]P_{\chi(x,t')} = \sum_j [|c(\psi, a_j)|e^{i\alpha(j)}/\langle\Phi_j(x, t')|\psi(x, t)\rangle]P_{\Phi_j(x, t')}$$

of preparation, out of the studied state vector $|\psi(x, t)\rangle$, of the superposition state vector

$$|\chi(x, t')\rangle = \sum_j |c(\psi, a_j)|e^{i\alpha(j)}|\Phi_j(x, t')\rangle, \qquad t' > t$$

where (a) $|\Phi_j(x, t')\rangle$, for any j, is a normed eigendifferential corresponding to an eigenvector $|u_j(x)\rangle$ of A, (b) the coefficients of linear combination reproduce the real parts $|c(\psi, a_j)|$ of the expansion coefficients $c(\psi, t, a_j)$ from the spectral decomposition $|\psi(x, t)\rangle = \sum_j c(\psi, t, a_j)|u_j(x)\rangle$ of the studied state vector $|\psi(x, t)\rangle$ on the basis of eigenvectors $|u_j(x)\rangle$ of A, the factors $e^{i(\alpha)j}$ being arbitrary (in particular these factors can reproduce those from the $c(\psi, t, a_j)$, or, alternatively, they can be all set equal to 1, thus introducing a superposition with real coefficients); and (c) the spatial domains $\Delta_j(\chi, t')$ where the presence probabilities corresponding to the state vectors $|\Phi_j(x, t')\rangle$ are not practically zero become mutually disjoint up to an approximation that can be improved without limitation by increasing t'.

This condition requires H_A such that out of the studied state vector $|\psi(x, t)\rangle$ there shall materialize approximately in the *physical* space, at times $t' > t$, the—abstract—spectral decomposition of $|\psi(x, t)\rangle$ on the basis of eigenvectors of A.

The condition CH_A, if it is realized, entails the following theorem.

Measurement Theorem MT. The event "registration for $|\psi(x, t)\rangle$ of an eigenvalue a_j of A" can be represented by the event ["registration for $|\chi(x, t')\rangle$ of the presence inside the domain $\Delta_j(\chi, t')$"] $\approx [x \in \Delta_j(\chi, t')]$, in the following sense. The numerical equality

$$\pi(\psi, a_j) = \pi(x \in \Delta_j(\chi, t'))$$

where $\pi(\psi, a_j)$ and $\pi(x \in \Delta_j(\chi, t'))$ are, respectively, the quantum mechanical probabilities of the first and the second event specified above, is realized with an arbitrarily improvable accuracy for any measurement evolution M_A of the first kind.

Proof. Consider the superposition state vector

$$|\chi(x, t')\rangle = \sum_j |c(\psi, a_j)| e^{i\alpha_j} |\Phi_j(x, t')\rangle, \qquad t' > t$$

as defined in CH_A. At any individual space point x we have for $|\chi(x, t')\rangle$ a presence probability which (at most) is reduced to only one term

$$\pi(x, \chi) = |\chi(x, t')|^2 = |c(\psi, a_q)| e^{i\alpha_q} \Phi_q(x, t')|^2 = |c(\psi, a_q)|^2 |\Phi_q(x, t')|^2$$

where the index q designates, among all the disjoint spatial domains $\Delta_j(\chi, t')$, the one to which the considered point x belongs. Then the *total* quantum mechanical presence probability inside the domain $\Delta_q(\chi, t')$ is, from the expression of $\pi(x, \chi)$ and because of the norm 1 of the $|\Phi_j(x, t')\rangle$,

$$\pi(x \in \Delta_q(\chi, t')) = \int_{\Delta_q} |\chi(x, t')|^2 \, dx = |c(\psi, a_q)|^2 \int |\Phi_q(x, t')|^2 \, dx = |c(\psi, a_q)|^2$$

which by the postulate (2) is also the quantum mechanical probability for the realization of the eigenvalue a_q. This is true only approximately but with an accuracy which according to CH_A can be improved arbitrarily by increasing t', i.e., by improving the mutual disjunction of the spatial domains $\Delta_j(\chi, t')$ and so the mutual orthogonality of any two distinct state vectors $|\Phi_j(x, t')\rangle$ and $|\Phi_k(x, t')\rangle$. So, with an arbitrarily improvable accuracy, we have indeed

$$\pi(x \in \Delta_q(\chi, t')) = |c(\psi(t), a_q)|^2 = \pi(\psi(t), a_q), \qquad t' > t \quad \blacksquare$$

This proof, trivial as it is, establishes a crucial *connection* between the two fundamentally *distinct* concepts of spectral decomposition and of superposition of states. More, in fact. It establishes for the general quantum mechanical predictional *postulate* (2)

$$\pi(\psi, a_j) = |\langle u_j | \psi \rangle|^2 = |\langle u_j | \psi \rangle|^2$$

an "explanation" *deduced* from the condition CH_A and the particular acceptance of (2) concerning exclusively the position observable, $\pi(\psi, x) = |\psi(x)|^2$. And notice that the deduction is founded upon the *distinction* between spectral decompositions of one state vector and superpositions of several state vectors.

Via the condition CH_A and the theorem MT the spectral decomposition of the studied state vector $|\psi(x, t)\rangle$ with respect to the eigenvectors of a measured observable A appears as only an *abstract conceptual prefiguration* of the superposition state $|\chi(x, t')\rangle$ actually prepared in the physical space, at later times $t' > t$, by the quantum mechanical operator H_A of measurement evolutions M_A.

By a rotation inside the Hilbert space of the system the measurement propagator H_A brings asymptotically the conceptual spectral decomposition, with respect to the eigenkets of A, of the studied state vector $|\psi(x, t)\rangle$, down onto the physical space. The *abstract* "disjunction" represented by the spectral decomposition $|\psi(x, t)\rangle = \sum_j c(\psi, t, a_j)|u_j(x)\rangle$ distinguishes inside $|\psi(x, t)\rangle$ between the elements of a family of mutually exclusive "how's" represented by eigenvectors $|u_j\rangle$, no matter *where* in space-time, since $\langle u_k|u_j\rangle = 0$ for $j \neq k$, but the $|u_j\rangle$ are time independent and in general distinct $|u_j\rangle$ do not possess disjoint spatial supports. The measurement propagator H_A transposes this abstract disjunction into a "disjunction" in the physical space, represented by the superposition state vector

$$|\chi(x, t')\rangle = \sum_j |c(\psi, a_j)| e^{i\alpha_j} |\Phi_j(x, t')\rangle, \qquad t' > t$$

that distinguishes between the elements of a family of mutually exclusive "where's," the $\Delta_j(\chi, t')$, while *how* is that what populates the disjoint spatial domains $\Delta_j(\chi, t')$ is devoid of pragmatic significance: With respect to the pair of qualifications how–where, the initial situation and the final one are *opposed.*

6. CONCLUSION

We have constructed an integrated view concerning the probabilistic organization of the quantum mechanical formalism. This view brings in four hierarchically connected descriptional levels:

1. The elementary quantum mechanical chain experiments (eqmce).
2. The basic probability chains (1'), (5), which are metastructures with respect to the elementary quantum mechanical chain experiments.
3. The probability trees of a state preparation $\mathcal{T}(P(\psi_0), |\psi\rangle)$, which are metastructures with respect to the basic probability chains (1'), (5).

4. Linear superpositions of probability trees which are metastructures with respect to the probability trees, namely compositions of several entire probability trees entailed by the principle of superposition (we do not mention the quantum mechanical algorithms representing successive measurements which, by use of the projection postulate, identify confusingly the preparable states of a microsystem and the eigenfunctions of an observable).

The integrated view concerning the probabilistic organization of quantum mechanics has acted as an instrument for critical analyses and for constructive developments.

The quantum mechanical calculi as well as the verbal accompaniments of these convey only very mutilated indications concerning the underlying probabilistic organization of the formalism. Vectors, operators, equations, probability measures, and operational definitions of measurements are manipulated according to algorithms. But the more global concepts of an elementary quantum mechanical chain experiment, of a random phenomenon (4), of a basic probability chain (1′), (5), and of a probability tree $\mathcal{T}(P(\psi_0), \psi\rangle)$, with their formal features and their specific semantic contents, seem to have remained not perceived. Not even the algorithmic shadow (1) of only an isolated basic probability chain (1′), (5), has been clearly recognized as a probabilistic whole. *A fortiori*, the distinction between formal entities and factual entities remained so dispersed and so vague that the central connecting role of the identities (7) has not been realized fully. This, no doubt, is due to the particular complexity of the random phenomena studied in quantum mechanics and to the unusual potential-actualization-actualized nature of the roots of the elementary events produced by these. The conjunction of these two characters acted as a barrier. We have overcome this barrier by a systematic reference to the basic concepts of the abstract theory of probabilities, by *an explicit specification of the cognitive operations by which the "observer," the "conceptor," produces the entities to be qualified (quantum mechanical states) and the processes of qualification of these (measurement evolutions)*, and by taking into account systematically *the space-time aspects of all the phenomena involved.*

This same sort of approach, resumed on a quite general level, has led us to a "general method of relativized conceptualization" (Mugur-Schächter, 1992d). This general method—a genuine "epistemic syntax", permitted us to return reflexively upon quantum mechanics wherefrom it stems and to further decode its semantic conducts and sharpen its algorithms (Mugur-Schächter, 1992c). Most important perhaps, it permitted us to clearly define the conceptual status of the quantum theory and to progress toward a model unifying quantum mechanics and relativity (Mugur-Schächter, 1992c, 1992f).

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